ISRAEL JOURNAL OF MATHEMATICS 106 (1998), 79–92

A DERIVATION HNN CONSTRUCTION FOR LIE ALGEBRAS

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ABSTRACT

In this paper, we define HNN extensions for Lie algebras. Given a Lie algebra L with a subalgebra A and a derivation d: $A \to L$, the HNN extension contains L and d extends to an inner derivation of it. We then use it to prove some results analogous to results in group theory.

1. Introduction

The HNN construction in group theory is well known. Let us briefly describe it. If A is a subgroup of a group G and $t \in G$, then the mapping $a \mapsto t^{-1}at$ is an isomorphism between the two subgroups A and $t^{-1}At$ of G. The HNN construction tries to reverse the view. Given a group G with an isomorphism ϕ between two of its subgroups A and B, we seek an extension H of G with an element $t \in H$, such that $t^{-1}at = \phi(a)$ for every $a \in A$. We are naturally led to consider the group H defined by the presentation

$$H = \langle G, t \mid t^{-1}at = \phi(a) \text{ for all } a \in A \rangle$$

and it emerges (for example, by using Britton's lemma on the form of reduced words in H) that G is in fact embedded in H.

We wish to perform a similar construction in Lie algebras. Attempts to define Lie HNN extensions by trying to realize an isomorphism between two subalgebras by means of an inner automorphism may be shown to fail in general (for a

Received January 5, 1997

definition of inner automorphisms and an explanation of the reasons of failure, see [3]). We proceed in a different path. If A is a subalgebra of the Lie algebra L (over a field k) and $t \in L$, then the map $a \overset{d}{\mapsto} [t, a]$ is k-linear and satisfies the derivation identity $d([a, b]) = [a, d(b)] + [d(a), b]$. As before, consider now a Lie algebra L with a subalgebra A and a derivation $d\colon A \to L$. We seek an extension H of L having an element $t \in H$, such that $[t, a] = d(a)$ for every $a \in A$. We shall define H in the natural way, provide a normal form for the elements of H (using Širšov's composition lemma) and deduce that L is embedded in H. We then carry some of the known group-theoretic applications of HNN extensions to Lie algebras. Results that involve torsion issues do not have Lie algebra analogues.

In section 2 we state Širšov's composition lemma. In section 3 we give a presentation for the HNN extension. In section 4 we show that the presentation is complete under compositions, so that we may apply Širšov's composition lemma to get a normal form for elements in the HNN extension in section 5. We then use the HNN extension to obtain results analogous to known results in group theory. In section 6, we recall two alternative ways to construct the HNN extension in group theory and propose Lie algebra analogues. In section 7 we prove that every Lie algebra of finite or countable dimension can be embedded in a Lie algebra with two generators and the same number of relations. In section 8 we prove that Markov properties of finitely presented Lie algebras are undecidable. Finally, in section 9 we show that finitely presented soluble Lie algebras are HNN extensions of a special form.

ACKNOWLEDGEMENT: I wish to thank Shmuel Rosset and Joseph Abarbanel for providing the motivation for the construction and for useful discussions.

2. The composition lemma

As mentioned, our proofs lean on Širšov's composition lemma. This lemma is the Lie algebra analogue of Bergman's diamond lemma [1], but it does not follow from it in any obvious way. In this section, we wish to define the relevant definitions and give a statement of the lemma. We follow [4] in this exposition (proofs may be found there).

Let X be an ordered alphabet. We order the words on X (which are the elements of the free monoid on X) lexicographically, such that initial subwords of a word are greater than the word itself. A **Lyndon word** is a word greater than any cyclic permutation of itself. For a non-associative word q (an element of the free magma on X which we shall also call a tree), we denote by $\overline{q}$ the associative word obtained by deleting the parentheses. It is called the **support** of q. We proceed and define **Lyndon trees** (in the free magma) by induction on the length. The elements of X are Lyndon trees and a tree $q = q_1q_2$ of degree ≥ 2 is a Lyndon tree if q_1 and q_2 are Lyndon trees, $\overline{q_1} > \overline{q_2}$, and if $q_1 = q'q''$ is of degree ≥ 2, then $\overline{q''} \leq \overline{q_2}$.

Given these definitions, every Lyndon word has a unique arrangement of parentheses making it a Lyndon tree and the support of every Lyndon tree is a Lyndon word. The set of Lyndon trees is a Hall set (see [7]), so that the associated Lie elements (obtained by interpreting the parentheses as Lie brackets) give a linear basis of the free Lie algebra $L(X)$.

Given an element $f \in L(X)$, express it as a linear combination of Lyndon trees. The **leading term** of f (denoted by $\tilde{f}$) is the largest Lyndon tree of all the trees of largest degree which appear with a non-zero coefficient in f. Given a Lyndon word s and disjoint subwords $f_1, \ldots, f_k$ which are Lyndon words, we can extend the standard arrangement of parentheses in all of the f_i's to an arrangement of parentheses in s, such that the support of the leading term of the corresponding Lie element is s.

We now proceed to define compositions. Suppose $f, g \in L(X)$, such that $\overline{\tilde{f}} = f_1h$, $\overline{\tilde{g}} = hg_1$ where h is a non-empty word and the coefficients in f and g of the leading terms are 1. Then f_1hg_1 is a Lyndon word. Take the parentheses in $\tilde{f}$ (with support f_1h) and extend them to parentheses in f_1hg_1 as in the last paragraph. If we substitute f for $\tilde{f}$, we get an element f_* (it is not unique, because there may exist several ways of extending the parentheses). Performing a similar thing on the subword hg_1 of f_1hg_1, we get an element g_*. The element $f_* - g_*$ is called a **first order composition** of f and g with respect to the subword h and is denoted $(f, g; h)$.

To define second order compositions, let $f, g \in L(X)$ be such that $\overline{\tilde{g}}$ is a subword of $\overline{\tilde{f}}$. We extend the parentheses of $\tilde{g}$ to $\overline{\tilde{f}}$ and substitute g for $\tilde{g}$. We get an element f_* and $f - f_*$ is called a **second order composition** of f and g and denoted (f, g).

Let now S be a subset of $L(X)$ and assume that the coefficients of the leading terms of all the elements of S are equal to 1 and that there are no second order compositions between elements of S. Assume also that for every $f, g \in S$ with a first order composition (with $\overline{\tilde{f}} = f_1h$ and $\overline{\tilde{g}} = hg_1$), we can write $(f, g; h) = \sum f_i$, such that each f_i is in the ideal generated by some $s_i \in S$ and $\overline{\tilde{f_i}} < f_1hg_1$. If all these assumptions hold, then S is said to be **complete under compositions**. The following is the composition lemma.

THEOREM 2.1: *Let $L(X)$ be the free Lie algebra on X and I the ideal generated by a set S complete under compositions. Then a non-zero $f \in L(X)$ is in I only if the support of the leading term of f contains a subword of the form $\bar{\bar{s}}$ for some $s \in S$.*

From the composition lemma one can deduce

COROLLARY 2.2: *Under the same notations, let B be the set of all Lyndon trees, whose support does not contain a subword of the form $\bar{\bar{s}}$ with $s \in S$. Then the image of B is a linear basis of $L(X)/I$.*

3. The construction

Let L be a Lie algebra over a field k and let A be a subalgebra of L. A **derivation** is a k-linear map $d\colon A \to L$, which satisfies the equation $d([a,b]) = [a, d(b)] + [d(a), b]$ whenever $a, b \in A$. Note that, contrary to common practice, we do not assume that the derivation is defined on the whole of L.

Suppose now that we have a derivation $d\colon A \to L$. We define the associated HNN extension to be the Lie algebra given by the presentation

$$H = \langle L, t \mid [t, a] = d(a) \text{ for all } a \in A \rangle.$$

By this we mean that we augment any presentation of L by adding a new generating letter t (called the **stable letter**) and, for each element of A, we add the relation $[t, a] = d(a)$ (where a and $d(a)$, being elements of L, are expressed in some way using the generators of L).

We mention the two extreme cases. If $A = L$, then d is a derivation of L and H is then the semi-direct product of L with a one-dimensional Lie algebra which acts on L via (multiples of) d. If $A = 0$, then H is the free product of L with a one-dimensional Lie algebra.

In the following, we shall give an equivalent presentation and prove that it is complete under compositions. Using Širšov's composition lemma, we shall deduce a normal form for elements of this HNN construction and from this normal form we shall deduce that L is embedded in H.

In order to give a new presentation for H, we choose a k-linear basis of L that includes a basis of A. We denote it by X and the subset corresponding to A by B. Arbitrary elements of X will be denoted by the letters u, v, x, y and z, whereas by the letters a and b we shall denote elements of B. Let the multiplication table of L be given by the structure constants

$$[x, y] = \sum_v \alpha_{xy}^v v$$

for $x, y \in X$. The fact that A is a subalgebra is reflected by the condition

$$\alpha_{ab}^{v} = 0 \quad \text{for } a, b \in B \text{ and } v \notin B. \tag{1}$$

Let the derivation d be given by the equations

$$d(a) = \sum_{v} \beta_{a}^{v} v$$

for $a \in B$. It is now clear that H has the following equivalent presentation (we use the k-linearity of the multiplication and of d):

$$\begin{aligned} H = \langle X, t \mid [x, y] &= \sum_{v} \alpha_{xy}^{v} v, \quad x, y \in X, x > y, \\ [t, a] &= \sum_{v} \beta_{a}^{v} v, \quad a \in B \rangle, \end{aligned} \tag{2}$$

where we have totally ordered X in some way.

4. Completeness under compositions

Before we prove that the presentation (2) is complete under compositions, we wish to see what relations hold between the α's and the β's. One can easily check that Jacobi's identity is reflected in the equations

$$\sum_{v} (\alpha_{xy}^{v} \alpha_{vz}^{u} + \alpha_{yz}^{v} \alpha_{vx}^{u} + \alpha_{zx}^{v} \alpha_{vy}^{u}) = 0 \tag{3}$$

for every $x, y, z, u \in X$. In addition, $\alpha_{xx}^{u} = 0$ and $\alpha_{xy}^{u} = -\alpha_{yx}^{u}$. The condition $d([a, b]) = [a, d(b)] + [d(a), b]$ is reflected by the equations

$$\sum_{v} \alpha_{ab}^{v} \beta_{v}^{u} = \sum_{v} (\beta_{b}^{v} \alpha_{av}^{u} + \beta_{a}^{v} \alpha_{vb}^{u}) \tag{4}$$

for every $a, b \in B$ and $u \in X$. Note that β_{v}^{u} is defined, whenever it is needed, because if $v \notin B$, we have $\alpha_{ab}^{v} = 0$ (by (1)).

For convenience, we denote the relation

$$[x, y] - \sum_{v} \alpha_{xy}^{v} v$$

by r_{xy} and the relation

$$[t, a] - \sum_{v} \beta_{a}^{v} v$$

by s_a. Though in the presentation (2) we have only taken the relation r_{xy} when $x > y$, we see that, since $r_{xx} = 0$ and $r_{xy} = -r_{yx}$ for $x < y$, there is no harm in considering r_{xy} as a relation for arbitrary x and y in X. We order X so that the elements of B are smaller than the elements of $X \setminus B$ and the stable letter t is the largest letter. Then it is clear that the only possible compositions are first order (overlap) compositions corresponding to the words xyz with $x, y, z \in X$ and $x > y > z$ and to the words tab with $a, b \in B$ and $a > b$ (we use here the property of the order that an element of X smaller than an element of B is an element of B).

We calculate the composition corresponding to the word xyz with $x, y, z \in X$ and $x > y > z$:

$$\begin{aligned}([x,y],[y,z];y) &= [r_{xy}, z] - [x, r_{yz}] \\ &= [[x,y] - \sum \alpha_{xy}^{v} v, z] - [x, [y,z] - \sum \alpha_{yz}^{v} v] \\ &= [[x,y],z] - \sum \alpha_{xy}^{v}[v,z] - [x,[y,z]] + \sum \alpha_{yz}^{v}[x,v].\end{aligned}$$

Using Jacobi's identity this equals

$$\begin{aligned}=&[[x,z],y] - \sum \alpha_{xy}^{v}([v,z] - \sum \alpha_{vz}^{u} u) \\ &+ \sum \alpha_{yz}^{v}([x,v] - \sum \alpha_{xv}^{u} u) - \sum \alpha_{xy}^{v}\alpha_{vz}^{u} u + \sum \alpha_{yz}^{v}\alpha_{xv}^{u} u \\ =&[[x,z] - \sum \alpha_{xz}^{v} v, y] + \sum \alpha_{xz}^{v}([v,y] - \sum \alpha_{vy}^{u} u) \\ &- \sum \alpha_{xy}^{v} r_{vz} + \sum \alpha_{yz}^{v} r_{xv} \\ &- \sum_{u}(\sum \alpha_{xy}^{v}\alpha_{vz}^{u} + \sum \alpha_{yz}^{v}\alpha_{vx}^{u} + \sum \alpha_{zx}^{v}\alpha_{vy}^{u})u,\end{aligned}$$

and using (3)

$$= [r_{xz}, y] + \sum \alpha_{xz}^{v} r_{vy} - \sum \alpha_{xy}^{v} r_{vz} + \sum \alpha_{yz}^{v} r_{xv}.$$

We thus see that the composition $([x,y],[y,z];y)$ is generated by relations with support less than xyz (the largest is xzy, which is less than xyz, because $y > z$). The above calculation is standard (it appears as lemma 3.3.6 in [4]). We include it here for completeness and because it resembles the next one.

Let us proceed with the calculation of the other type of composition, corresponding to the word tab with $a, b \in B$ and $a > b$:

$$\begin{aligned}([t,a],[a,b];a) &= [s_a, b] - [t, r_{ab}] \\ &= [[t,a] - \sum \beta_{a}^{v} v, b] - [t, [a,b] - \sum \alpha_{ab}^{v} v]] \\ &= [[t,a],b] - \sum \beta_{a}^{v}[v,b] - [t,[a,b]] + \sum \alpha_{ab}^{v}[t,v].\end{aligned}$$

Noting that only v's from B appear in the second sum, because of (1),

$$
\begin{aligned}
=&[[t,b],a]-\sum\beta_a^v([v,b]-\sum\alpha_{vb}^u u)\\
&+\sum\alpha_{ab}^v([t,v]-\sum\beta_v^u u)-\sum\beta_a^v\alpha_{vb}^u u+\sum\alpha_{ab}^v\beta_v^u u\\
=&[[t,b]-\sum\beta_b^v v,a]+\sum\beta_b^v([v,a]-\sum\alpha_{va}^u u)\\
&-\sum\beta_a^v r_{vb}+\sum\alpha_{ab}^v s_v\\
&+\sum_u(\sum\alpha_{ab}^v\beta_v^u-\sum\beta_b^v\alpha_{av}^u-\sum\beta_a^v\alpha_{vb}^u)u
\end{aligned}
$$

now using (4)

$$=[s_b,a]+\sum\beta_b^v r_{va}-\sum\beta_a^v r_{vb}+\sum\alpha_{ab}^v s_v.$$

Again, the largest support is tba which is less than tab.

We have checked both kinds of compositions and see that the presentation is complete under compositions.

5. Normal form

From the completeness under compositions of the presentation (2) and from Širšov's composition lemma, we get a normal form for the elements of the HNN extension.

THEOREM 5.1: *A k-linear basis for H is given by all the Lyndon words on $X\cup\{t\}$ which do not contain subwords of the form xy with $x,y\in X$ and $x>y$ or of the form ta with $a\in B$.*

Using this normal form, we deduce

COROLLARY 5.2: *L is embedded in H.*

Proof: All the elements of X are words in normal form. ∎

Another corollary which will be used later is the following:

COROLLARY 5.3: *Let x be an element in L but not in A. Then t and x are a free basis for the subalgebra generated by them.*

Proof: One may choose the basis X such that x is in X (and by assumption not in B). One can clearly see that all the Lyndon words on the letters x and t are in normal form, so that the subalgebra of H generated by t and x is freely generated by them. ∎

The last corollary can be stated in a less specific way:

COROLLARY 5.4: *Assume that $A \neq L$. Then H includes a free Lie subalgebra of rank* 2.

6. Alternative constructions

In group theory there are alternative ways to achieve the HNN construction, see [8, page 8]. Similar procedures exist also for Lie algebras and will be described in this section. Since we do not have specific applications for these alternative constructions, the proofs (which rely on our original construction) are omitted.

6.1 THE FIRST GROUP CONSTRUCTION. Let A be a subgroup of G with a monomorphism ϕ from A into G. If we wish to extend ϕ to the whole of G in a universal manner, we take the free product of two copies of G amalgamated along A, such that A is embedded in the first copy by the usual inclusion and in the second copy through ϕ. The identity mapping from the first copy of G to the second is the desired extension. Let us call this group G_1 and the extension ϕ_1. We may now repeat the process and embed G_1 in a group G_2 in which ϕ_1 extends to ϕ_2 defined on G_1. Repeating in a similar manner, we get an increasing sequence of groups G_n, with monomorphisms ϕ_n from G_{n-1} into G_n. Taking K as the union of the G_n, we get a group K with a monomorphism Φ from K into itself. If Φ is onto, we get an automorphism of K. If not (this situation will not have to be dealt with in the Lie case), we may take the inverse of Φ defined on the image of Φ and extend it in the above manner and then alternatingly extend the monomorphism and its inverse. Taking the union, we get an automorphism. Now we may take the semi-direct product of the group with an infinite cyclic group generated by t, with t acting as the automorphism. This is the HNN extension.

6.2 UNIVERSAL EXTENSION OF A DERIVATION AND THE LIE CONSTRUCTION. Let A and B be subalgebras of a Lie algebra L such that B contains A and let $d\colon A \to L$ be a derivation. We wish to embed L in a Lie algebra L' with a derivation $d'\colon B \to L'$ which extends d. Introduce new elements d_b indexed by the elements of B and take the Lie algebra generated by L and the new elements d_b subject to the relations of L and the following additional relations:

1. $d_{b+b'} = d_b + d_{b'}$ for all $b, b' \in B$.
2. $d_{\alpha b} = \alpha d_b$ for all scalars α and $b \in B$.
3. $d_{[b,b']} = [b, d_{b'}] + [d_b, b']$ for all $b, b' \in B$.

4. $d_a = d(a)$ for all $a \in A$.

If we show that L is embedded in L', then it is clear that the mapping $d'\colon B \to L'$ defined by $d'(b) = d_b$ is a derivation which extends d. In order to show that L is embedded in L', it suffices to show that there exists a Lie algebra L'' containing L with a derivation $d''\colon B \to L''$ which extends d, for if that is the case then it is clear that the subalgebra of L'' generated by L and $d''(B)$ is a quotient of L' and we may deduce that L is embedded in L''. For that we may take L'' to be the HNN extension defined by L, A and d, because we have shown that L is embedded in L'' and the mapping $\operatorname{ad} t$ extends d to the whole of L''.

Suppose now that $d\colon A \to L$ is a derivation. Use the construction above (with $B = L$) to embed L in a Lie algebra L_1 with a derivation $d_1\colon L \to L_1$ which extends d. Repeat and embed L_1 in a Lie algebra L_2 with a derivation $d_2\colon L_1 \to L_2$ which extends d_1. Repeating we get an increasing sequence of Lie algebras L_n with derivations $d_n\colon L_{n-1} \to L_n$, such that d_n extends d_{n-1}. Taking M as the union, we get a Lie algebra with a derivation $D\colon M \to M$. Now take the semi-direct product H of M with a one-dimensional Lie algebra spanned by t, with t acting on M via D. This H is the HNN extension defined by A, L and d.

6.3 THE SECOND GROUP CONSTRUCTION. Again, we are given A, G and ϕ. In the first construction we gradually extended ϕ until we received a group containing G with an extension of ϕ to an automorphism. We then took the semi-direct product with an infinite cyclic group. Now, we shall describe a slightly different way to perform the extension. Take an infinite number of copies of G, G_n, indexed by the integers. Take the free product of the G_n with amalgamation that identifies the copy of A in G_n with the copy of $\phi(A)$ in G_{n+1} using ϕ. The mapping which maps G_n to G_{n+1} by the identity is an automorphism which extends ϕ (if we identify G_0 with G).

6.4 THE LIE CONSTRUCTION. Given A, L and d, we wish to embed L in a Lie algebra in which d extends to a derivation. Take the Lie algebra generated by elements $d_{n,l}$ with n running over the non-negative integers and l over L and subject to the following relations:

1. On the elements $d_{0,l}$ impose the relations of L.
2. $d_{n,l+l'} = d_{n,l} + d_{n,l'}$.
3. $d_{n,\alpha l} = \alpha d_{n,l}$.

4. $d_{n,[l,l']} = \sum_{i=0}^{n} \binom{n}{i} [d_{i,l}, d_{n-i,l'}]$.

5. $d_{1,a} = d(a)$ for all $a \in A$.

In this Lie algebra one can define a derivation by mapping $d_{n,l}$ to $d_{n+1,l}$ and then (as in the construction of 6.2) take the semi-direct product with a one-dimensional algebra to get the HNN extension.

7. An embedding theorem

We now prove an analogue of an embedding theorem known to hold in groups (see e.g. [6]). We need a lemma.

LEMMA 7.1: *Let A be a Lie algebra and L a subalgebra which is free on a subset X. Then every map $d\colon X \to A$ can be uniquely extended to a derivation from L to A.*

Proof: Uniqueness is clear.

Let A_1 be an abelian Lie algebra with underlying linear space A. Define $h\colon L \to \mathrm{Der}(A_1)$ by $h(l)(a) = [l, a]$. It is clear that $h(l)$ is indeed in $\mathrm{Der}(A_1)$, since any linear endomorphism of A_1 is a derivation (A_1 is abelian). h is a Lie homomorphism by the usual calculation (using Jacobi's identity).

Define $B = L \ltimes A_1$ (L acts on A_1 through h) and a Lie homomorphism $f\colon L \to B$ by $f(x) = (x, d(x))$ for every $x \in X$. Write $f(l) = (u(l), D(l))$. Then

$$\begin{aligned}(u([l_1,l_2]), D([l_1,l_2])) =& f([l_1,l_2]) = [(u(l_1), D(l_1)), (u(l_2), D(l_2))] \\ =& ([u(l_1), u(l_2)], [u(l_1), D(l_2)] - [u(l_2), D(l_1)]).\end{aligned}$$

We conclude $u([l_1,l_2]) = [u(l_1), u(l_2)]$, which implies (since $u(x) = x$ for $x \in X$) $u(l) = l$ for every $l \in L$. Using this,

$$D([l_1,l_2]) = [l_1, D(l_2)] - [l_2, D(l_1)] = [l_1, D(l_2)] + [D(l_1), l_2],$$

so that D is a derivation which extends d. ∎

THEOREM 7.2: *Every finite- or countable-dimensional Lie algebra A can be embedded in a Lie algebra H generated by two elements. If A has a presentation with n relations, so does H.*

Proof: Let

$$A = \langle c_1, c_2, \dots \mid R \rangle$$

be a presentation of L using countably many generators. Define $A_1 = A * L(a,b)$, the free product of A with a free Lie algebra of rank 2. The elements $a, ba, b^2a, \ldots, b^na, \ldots$ (by b^na we mean $[b,\ldots,[b,a]\ldots]$ where b appears n times) are a basis for a free Lie subalgebra of $L(a,b)$ (by Lazard's elimination theorem, see [7]). Therefore, using the lemma, there exists a derivation from that subalgebra to A_1 that maps $a \mapsto b$, $b^na \mapsto c_n$. Consider the corresponding HNN extension

$$H = \langle A_1, t \mid [t, b^na] = c_n, [t,a] = b \rangle.$$

Then A is embedded in H and H is generated by t and a. Note that all the HNN relations can be eliminated by Tietze transformations (eliminating the generators c_n and b), so that if R has n relations, we can get a presentation for H with n relations. ∎

Note that if the algebra is recursively presented, then one can effectively compute a presentation for the embedding algebra.

8. Undecidability of Markov properties

Let P be a property of finitely presented Lie algebras preserved under isomorphism. P is called **Markov** if there exists a finitely presented Lie algebra L_1 satisfying P and a finitely presented Lie algebra L_2 which cannot be embedded in a finitely presented Lie algebra satisfying P. Every non-trivial property P preserved by taking subalgebras is Markov (e.g. being zero, abelian, nilpotent, soluble, free). The following is an exact analogue of a theorem in group theory (see [6]). The proof is based on the group theoretic proof in [6]. For another proof, see [4].

THEOREM 8.1: *Let P be a Markov property of finitely presented Lie algebras. Then there exists no algorithm that determines whether a finitely presented Lie algebra satisfies P.*

Proof: Let L be any finitely presented Lie algebra with unsolvable word problem (for existence of such algebras, see [4]). Let w be any Lie expression in the generators of L. Consider the free product $L_2 * L$. It is finitely presented and, by the remark following Theorem 7.2, we can effectively write a finite presentation for a Lie algebra U generated by u_1 and u_2 in which $L_2 * L$ is embedded.

Now define

$$J = \langle U, y_1, y_2 \mid [y_1, u_1] = u_1, [y_2, u_2] = u_2 \rangle,$$
$$K = \langle J, z \mid [z, y_1] = y_1, [z, y_2] = y_2 \rangle,$$

$$Q = \langle r, s, t \mid [s, r] = r, [t, s] = s \rangle,$$
$$D_w = \langle K * Q \mid r = z, t = w \rangle,$$
$$E_w = D_w * L_1.$$

First of all, let us analyze Q. It is an HNN extension (with stable letter t) of the subalgebra generated by r and s over the base subalgebra spanned by s. The subalgebra generated by r and s is in itself an HNN extension (with stable letter s) of a one-dimensional algebra (spanned by r). In the subalgebra generated by r and s, r and s are linearly independent (using the normal form theorem 5.1). Therefore r is not in the base subalgebra for the second HNN extension and, using Corollary 5.3, we deduce that r and t freely generate a subalgebra of Q.

J clearly is an HNN extension (with stable letter y_2) of an HNN extension (with stable letter y_1) of U. y_1 is not in the base algebra for the HNN extension J, so by Corollary 5.3, the subalgebra generated by y_1 and y_2 is freely generated by them. We deduce by Lemma 7.1 that there exists a (unique) derivation that sends y_1 and y_2 to themselves so that K is an HNN extension (with stable letter z) of J.

Suppose that $w \neq 0$ in L. Using the normal form theorem 5.1 for the HNN extension J, U intersects the subalgebra generated by y_1 and y_2 only in 0. Therefore, w is not in the base subalgebra for the HNN extension K and, again using Corollary 5.3, we deduce that z and w freely generate a subalgebra of K.

To summarize, r and t freely generate a free Lie subalgebra of rank 2 of Q and (supposing $w \neq 0$ in L) z and w freely generate a free Lie subalgebra of rank 2 of K. We deduce that D_w is the free product of K and Q with these two free subalgebras amalgamated. We deduce that E_w contains a copy of L_2 and so E_w does not satisfy P.

Assume now that $w = 0$ in L. Then we consecutively deduce that $t = 0$ in D_w, $s = 0$ in Q, $r = 0$ in Q, $z = 0$ in D_w, $y_1 = y_2 = 0$ in K, $u_1 = u_2 = 0$ in J. This exhausts all the generators of D_w, so that E_w is isomorphic to L_1 and satisfies P.

We have shown that E_w satisfies P if and only if $w = 0$ in L. Since a finite presentation for E_w can be effectively computed given the expression for w, an algorithm for determining whether a finitely presented Lie algebra satisfies P could have been used to solve the word problem in L. We deduce that no such algorithm exists. ∎

9. Another application

The following is an analogue of a result of Bieri and Strebel ([2], we follow [9]).

THEOREM 9.1: *Let L be a finitely presented Lie algebra and let I be an ideal of L of co-dimension 1. Let a be an element in $L \setminus I$. Then there exist finitely generated subalgebras S and B of I with $S \subset B$ such that the restriction of* $\operatorname{ad} a$ *to S is a derivation $d\colon S \to B$ and such that the inclusion of B in L and the assignment $t \mapsto a$ induce an isomorphism*

$$\psi\colon \langle B, t \mid [t,s] = d(s) \text{ for all } s \in S\rangle \xrightarrow{\sim} L$$

between L and the HNN extension defined by B and d.

Proof: Every element of L is of the form $i + \alpha a$ with $i \in I, \alpha \in k$. Therefore one can find $b_1, \dots, b_f \in I$ such that $L = \langle b_1, \dots, b_f, a\rangle$. Let A be the free Lie algebra on $\{x_1, \dots, x_f, y\}$ and let π be the epimorphism $A \to L$ obtained by mapping $x_i \mapsto b_i$ (for $i = 1, \dots, f$) and $y \mapsto a$. Since L is finitely presented and A is finitely generated, the kernel R of π is finitely generated as an ideal, say by $r_1, \dots, r_l$. Let J be the (one co-dimensional) ideal of A generated by $x_1, \dots, x_f$. Then $R \subset J$ (since I is an ideal) and, by Lazard's elimination theorem ([5], see [7]), J is generated as a subalgebra by (and in fact is a free Lie algebra on) elements of the form $y^n x_i$. Since l is finite, there exists a natural number m such that all the r_j's are in

$$U = \langle y^r x_i, i = 1, \dots, f \text{ and } r = 0, \dots, m\rangle.$$

Now define $S = \langle a^r b_i,\ i = 1, \dots, f \text{ and } r = 0, \dots, m-1\rangle$ and $B = \langle a^r b_i,\ i = 1, \dots, f \text{ and } r = 0, \dots, m\rangle$. Then the restriction of $\operatorname{ad} a$ to S is a derivation $d\colon S \to B$ (the image is in B, since $\operatorname{ad} a$ sends the generators of S to B and by Jacobi's identity).

The inclusion of B in L and the assignment $t \mapsto a$ induce a homomorphism

$$\psi\colon \tilde{L} = \langle B, t \mid [t,s] = d(s) \text{ for all } s \in S\rangle \to L,$$

which is an epimorphism since $b_1, \dots, b_f \in B$. The assignments $x_i \mapsto b_i$, $y \mapsto t$ define a homomorphism $\chi\colon A \to \tilde{L}$. In $\tilde{L}$, $[t, a^r b_i] = a^{r+1} b_i$ for $r = 0, \dots, m-1$ and $i = 1, \dots, f$ (by the defining relations). It follows that $t^r b_i = a^r b_i$ in $\tilde{L}$ for $r = 0, \dots, m$ and $i = 1, \dots, f$. Therefore, χ maps the relator r_j, expressed in terms of the $y^r x_i$ in that range to the corresponding expression in terms of $a^r b_i$, which is equal to 0. Therefore, χ induces an epimorphism $A/R \twoheadrightarrow \tilde{L}$ and, since $\psi \circ \chi = \pi$, it follows that ψ is an isomorphism. ∎

We apply this in the soluble case (as in [2]).

COROLLARY 9.2: *Let L be a finitely presented soluble Lie algebra and I be an ideal of L of co-dimension* 1. *Then I is finitely generated as a subalgebra.*

Proof: By the previous theorem, L is an HNN extension over a finitely generated base subalgebra B contained in I. If (in the notations of the last theorem) S were a proper subalgebra of B, Corollay 5.4 would have provided us with a non-abelian free subalgebra of L. This is impossible, since L is soluble. We conclude that $S = B$. But then it is easy to see that L is a semi-direct product of B and a one-dimensional subalgebra, so B (which is contained in I) is of co-dimension 1 in L. This implies that $B = I$, so we conclude that I is finitely generated as a subalgebra. ∎

We note that every non-zero soluble Lie algebra has an ideal of co-dimension 1 (containing the derived subalgebra).

References

[1] G. M. Bergman, *The diamond lemma for ring theory,* Advances in Mathematics **29** (1978), 178–218.

[2] R. Bieri and R. Strebel, *Almost finitely presented soluble groups,* Commentarii Mathematici Helvetici **53** (1978), 258–278.

[3] L. A. Bokut', *Unsolvability of the equality problem and subalgebras of finitely presented Lie algebras,* Izvestiya Akademii Nauk SSSR, Seriya Matematicheskaya **36** (1972), 1173–1219 (Russian); English translation: Mathematics of the USSR—Izvestiya **6** (1972), 1153–1199.

[4] L. A. Bokut' and G. P. Kukin, *Algorithmic and Combinatorial Algebra,* Kluwer Academic Publishers, Dordrecht/Boston/London, 1994.

[5] M. Lazard, *Groupes, anneaux de Lie et problème de Burnside,* Istituto Matematico dell'Università di Roma, 1960.

[6] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory,* Springer, Berlin, 1977.

[7] C. Reutenauer, *Free Lie Algebras,* London Mathematical Society Monographs, New Series, Oxford Science Publications, Oxford, 1993.

[8] J.-P. Serre, *Trees,* Springer, Berlin, 1980.

[9] R. Strebel, *Finitely presented soluble groups,* in *Group Theory: Essays for Philip Hall* (K. W. Gruenberg and J. E. Roseblade, eds.), Academic Press, New York, 1984, pp. 257–314.